Monetary Policy, Bounded Rationality and Incomplete Markets*

Emmanuel Farhi Harvard University

Iván Werning MIT

September 2016

1 Introduction

This paper studies the effects of monetary policy in the presence of nominal rigidities exploring two realistic departures from standard benchmark models. First, we depart from the representative agent or complete market assumption by considering, instead, heterogeneous agents and financial market imperfections, including both incomplete insurance and borrowing constraints. Second, we depart from rational expectations, by adopting a particular form of bounded rationality. As we shall argue, our choice amongst the "wilderness" of options of non-rational expectations seems well suited to the economic and policy scenario that we shall focus on.

We consider changes in current and future interest rates and study their effect on aggregate output. In standard New-Keynesian models changes in future rates are as potent as changes in current rates, a property that some have labeled the 'forward-guidance puzzle'.[1] Although each of the two departures we consider may affect this property, we show that each deviation exerts only a moderate influence in isolation. The combined effect of both deviations, however, is potent and potentially affects the workings of monetary policy significantly. In other words, incomplete markets and level-k bounded rationality are *complements*. This highlights a more general point, that it is not always enough to investigate one deviation at a time from standard benchmark models.

*We are grateful to Mikel Petri, who provided outstanding research assistance.

[1]Given standard empirical identification challenges, which are only heightened when focusing on forward guidance shocks, relative to standard monetary shocks, one can interpret the term 'puzzle' as reflecting the property of standard models relative to a prior where monetary policy in the far future has smaller effects on current activity.

Our first deviation drops the representative agent and complete market assumptions. Our model is populated by heterogenous agents making consumption decisions subject to idiosyncratic shocks to income that cannot be insured. In addition, borrowing may also be limited. These frictions affect the capacity of households to smooth their consumption, potentially affecting the potency of forward guidance. Intuitively, if agents expect to be borrowing constrained in the near future, then changes in future interest rates should not greatly influence their current consumption decisions. This line of reasoning was put forth by McKay et al. (2015). However, as shown by Werning (2015), while incomplete markets always have an effect on the *level* of aggregate consumption, the way it affects its *sensitivity* to current and future interest rates is less clear. Indeed, this sensitivity is completely unchanged in some benchmark cases and may be enhanced in others. This implies that the power of forward guidance is not necessarily diminished by incomplete markets, at least not without adopting other auxiliary assumptions.[2] Here we adopt the benchmark cases that imply the neutral conclusion that incomplete markets have no effect on the sensitivity of aggregate consumption to interest rates.

Our second deviation drops the rational expectations assumption, adopting instead a form of bounded rational expectations introduced in macroeconomic settings by Woodford (2013) and García-Schmidt and Woodford (2015). The expectations we consider are based on a finite deductive procedure involving k iterations, which we refer to as "level-k thinking".

Under our adaptation, households are perfectly aware of the entire path of current and future interest rates, which are set by the monetary authority.[3] We are motivated by specific policy contexts, especially at the zero lower bound, where the intended interest rate path is directly and exhaustively communicated by the central bank; economic actors pay close attention to these announcements. In contrast, expectations for other endogenous variables are not under the direct control of the central bank and, thus, not directly announced and agents can only form beliefs about them indirectly. In our formulation, agents make an effort to think through the behavior of these variables, but stop short of achieving perfect foresight. This is motivated by two other considerations. First, the no-

[2] As highlighted in Werning (2015), two features that push to mitigate the impact of future interest rates relative to current interest rates on current aggregate consumption are: (i) procyclicality of income risk, making precautionary savings motives low during a recession; and (ii) countercyclicality of liquidity relative to income, making asset prices or lending fluctuate less than output. If one adopts the reverse assumptions, as a large literature does—so that recessions heighten risk, precautionary savings and are accompanied by large drops in asset prices or lending relative to GDP—then aggregate consumption becomes even more sensitive to future interest rates, relative to current interest rates.

[3] To simplify our model and allow for a non-linearized solution we consider cases where there is no ongoing aggregate uncertainty; all uncertainty has been resolved at $t = 0$, including any unexpected "MIT" shock. Thus, rational expectations is equivalent to perfect foresight.

tion that the status quo is a natural focal point to start to reason through the effects of the policy on future variables and that agents may be limited, or believe that others are limited, in performing this deduction. Second, we are interested in relatively unfamiliar scenarios, with interest rates near zero at an effective lower bound, where learning protocols that converge to rational expectations cannot be naturally invoked. Indeed, if agents recognize the uniqueness of the situation they may understand that past experience may be a poor guide. For these reasons, backward-looking data-based learning approaches to the formation of expectations may be inadequate in these circumstances. Our modeling of expectations is, instead, entirely forward-looking and deductive.

For simplicity, we consider an extreme form of nominal rigidity, where prices are completely rigid. This allows us to focus on real interest rate changes. It also implies that the only endogenous equilibrium variable that households must forecast is the level of aggregate income.[4] Indeed, we assume that in each period agents are perfectly aware of their current income and assets; moreover, agents have perfect foresight regarding the path for nominal interest rates (reflecting a credible announcement by the central bank). However, agents must form expectations about their future aggregate income. To do so they must forecast future aggregate output. We consider equilibria where these expectations satisfy a level-k iterative procedure that works as follows.

Level-1 thinking assumes that agents expect the path for future output to remain as in the original rational-expectations equilibrium before the announced change in the path of interest rates. Given current assets and income, individuals choose consumption and savings, reacting to the new interest rate path, using the status quo expectations for future income. In equilibrium, aggregate output equals aggregate consumption in each period. Thus, the economy is in general equilibrium, with agent demands their given expectations.

In the k-th deductive round, households take the path of future output to be the equilibrium path of output that obtains in the previous k round, etc. ad infinitum. We define level-k expectations to be the outcome after k such iterations. The solutions converges to the rational expectations solution when the number of rounds goes to infinity, $k \to \infty$. This process interpolates between a first round where the output effects monetary policy only reflect partial equilibrium effects, and rational expectations which also fully incorporate general equilibrium effects.

By itself, our level-k bounded rationality introduces both mitigation—the output ef-

[4]If prices were partially flexible, then households must additionally form expectations about future inflation. This affects the expected path for the real interest rate, for a given path for the nominal interest rate.

fects of interest rate changes—and horizon effects—the output effects of interest rate changes are more mitigated, the further in the future these interest rate changes take place. However, we show that these mitigation and horizon effects introduced by bounded rationality per se are relatively modest. Indeed, they may vanish when the interest rate is at zero.

Combining both departures, we show that the mitigation and horizon effects introduced by bounded rationality are much stronger in the presence of incomplete markets, even in cases where market incompleteness on its own would have no such effects. We conclude that the interaction of bounded rationality and incomplete markets is important, even if each element has modest effects in isolation.

We start in Section 2 by defining our different equilibrium concepts (temporary equilibrium, rational expectations equilibrium, level-k equilibrium) in a reduced form model for which the key primitive is a reduced-form aggregate consumption function. In Sections 3-5, we spell out complete models of individual behavior and show how they generate a reduced-form aggregate consumption function, so that we can apply the general definitions of Section 2.

We set up the complete markets case of a representative agent in Section 3. We then move on to study economies with incomplete markets in Sections 4-5. Putting our results together allows us to neatly characterize the interaction of bounded rationality and incomplete markets.

We unpack the separate contributions of the different features of incomplete markets: binding borrowing constraints and precautionary savings with uninsurable idiosyncratic income risk. To isolate the impact of occasionally-binding borrowing constraints, we consider first in Section 4 a perpetual youth overlapping generations model with annuities which we show can be reinterpreted as a model with occasionally binding borrowing constraints but without precautionary savings. We then put together occasionally-binding borrowing constraints and precautionary savings in an Aiyagari-Bewley model in Section 5. We show theoretically and quantitatively that a key determinant of the strength of the mitigation and horizon effects is the interaction of incomplete markets and bounded rationality. Detailed intuitions for these results are provided in the paper.

2 Level-k in a Simple Reduced-Form Model

We being by introducing the basic concepts of level-k equilibrium within a simplified model building on a reduced-form aggregate consumption function. Various explicit disaggregated models can be explicitly reduced to this formulation. For example, representative agent models, overlapping generations models, models with a fraction of permanent-income consumers and a fraction of hand-to-mouth consumers, and Aiyagari-Bewley models of heterogenous agents with income fluctuation and incomplete markets, all give rise to an aggregate consumption function of the form considered below. We will make this mapping explicit for several of these models in future sections.

2.1 Baseline Reduced-Form Model

We consider a simple model with one consumption good in every period and no investment. Time is discrete and the horizon is infinite with periods $t = 0, 1, \ldots$ We denote current and future real interest rates by $\{R_{t+s}\}$, and current and future aggregate income by $\{Y_{t+s}\}$, where s runs from 0 to ∞. We are interested in situations with nominal rigidities where the real interest rate is under the control of monetary authority. In particular, we focus for simplicity on the extreme case with perfectly rigid prices, where real interest rates equal nominal interest rates. Thus, we take the path of $\{R_{t+s}\}$ as given. Our goal is to solve for the equilibrium path of aggregate income $\{Y_{t+s}\}$.

Aggregate consumption function. We postulate an aggregate consumption function

$$C_t = C^*(\{R_{t+s}\}, Y_t, \{Y^e_{t+1+s}\}), \tag{1}$$

where $\{Y^e_{t+1+s}\}$ denotes future anticipated aggregate income.

The fact that the aggregate consumption function depends only on current and future interest rates, current income and future anticipated income is useful and merits brief discussion. With a representative agent such a formulation is straightforward, and we discuss that example below. Otherwise, the consumption function should be interpreted as performing an aggregation and consolidating any distributional effects, including solving out for wages and profits as a function of current Y_t. Implicitly we are also assuming there is no heterogeneity in beliefs about future income, $\{Y^e_{t+1+s}\}$, although one may extend the analysis to capture heterogeneity in beliefs.

In this formulation the consumption function is purely forward looking—it does not depend on the past or on any state variable that is affected by the past. This can accommodate various interesting and simple models, such as the representative agent, the perpetual youth overlapping generations model and certain simple models with heterogeneity (such as having a fraction of hand to mouth agents). It does not fit all situations, however. In the next subsection we provide an extension with a state variable that captures standard Aiyagari-Bewley models.

Temporary equilibria. We are interested in allowing for more general beliefs than rational expectations. We start by defining the notion of a temporary equilibrium, which takes as given a sequence of beliefs $\{Y_t^e\}$ and simply imposes that the goods market clear

$$Y_t = C_t. \tag{2}$$

Definition (Temporary equilibrium)**.** Given a sequence of beliefs $\{Y_t^e\}$, a temporary equilibrium is a sequence $\{R_t, Y_t\}$ satisfying (1) and (2) for all $t \geq 0$.

Start at some baseline temporary equilibrium $\{R_t, Y_t, Y_t^e\}$ and consider an new announced interest rate path $\{\hat{R}_t\}$ at $t = 0$. How does this affect the equilibrium? The answer depends on the adjustment of beliefs. We now describe two possible adjustments of beliefs: rational expectations and level-k thinking.

Rational expectations equilibria. A rational expectations equilibrium is a particular case of temporary equilibrium with the extra requirement of perfect foresight, i.e. that beliefs about future income coincide with actual future income

$$\{Y_t^e\} = \{Y_t\}. \tag{3}$$

.

Definition (Rational expectation equilibrium)**.** A rational expectations equilibrium (REE) is a sequence $\{R_t, Y_t, Y_t^e\}$ such that $\{R_t, Y_t\}$ is a temporary equilibrium given beliefs $\{Y_t^e\}$ and which satisfies perfect foresight (3) for all $t \geq 0$.

For notational convenience, we often denote a given REE by $\{R_t, Y_t\}$ instead of $\{R_t, Y_t, Y_t\}$.

Start at some baseline REE $\{R_t, Y_t\}$ and consider an new announced interest rate path $\{\hat{R}_t\}$ at $t = 0$. Under rational expectations, there is an issue about selection since there are typically several REEs for a given interest rate path $\{\hat{R}_t\}$. In our detailed applications, and for the considered interest rate paths, we will always be able to select a unique REE by imposing that the baseline and new REEs coincide in the long run:

$$\lim_{t \to \infty} \hat{Y}_t = \lim_{t \to \infty} Y_t.$$

From now on, we always use this selection.

Level-k equilibria. We now deviate from rational expectations and describe an alternative adjustment of expectations encapsulated in the notion of level-k equilibrium. We then introduce the notion of level-k equilibrium $\{R_t, \hat{Y}_t^k\}$ which specifies a sequence of beliefs $\{\hat{Y}_t^{e,k}\}$ indexed by k. As above, we start at some baseline REE $\{R_t, Y_t\}$, and consider a one-time unexpected shock change in the path for the interest rate $\{\hat{R}_t\}$ at $t = 0$.

The level-1 equilibrium $\{\hat{R}_t, \hat{Y}_t^1\}$ is a temporary equilibrium given beliefs $\{\hat{Y}_t^{e,1}\} = \{Y_t\}$ corresponding to the aggregate income path of the original REE. In other words, expectations for future aggregate income are unchanged after the announced change in interest rates and equal to the original REE path. For each each $t = 0, 1, \ldots$ $\hat{Y}_t^1$ can be computed as the following fixed point equation

$$\hat{Y}_t^1 = C^*(\{\hat{R}_{t+s}\}, \hat{Y}_t^1, \{Y_{t+1+s}\}).$$

The level-1 equilibrium captures a situation where agents take into account the new announced path for interest rates and observe present income, but do not adjust their expectations about future income. However, actual realized income is affected.

The level-2 equilibrium $\{\hat{R}_t, \hat{Y}_t^2\}$ is a temporary equilibrium given beliefs $\{\hat{Y}_t^{e,2}\} = \{\hat{Y}_t^1\}$ corresponding to the aggregate income path from level-1. For every $t \geq 0$, $\hat{Y}_t^2$ can be computed as the following fixed point equation

$$\hat{Y}_t^2 = C^*(\{\hat{R}_{t+s}\}, \hat{Y}_t^2, \{\hat{Y}_{t+1+s}^1\}).$$

Here agents update their beliefs to take into account that the change in aggregate spending (by all other agents) associated with level-1 thinking has an effect on aggregate income (and hence on their own income). In other words, level-2 thinking incorporates the general equilibrium effects of future income from level 1.

Continuing, the level-k equilibrium $\{\hat{R}_t, \hat{Y}_t^k\}$ is defined as a temporary equilibrium given beliefs $\{\hat{Y}_t^{e,k}\} = \{\hat{Y}_t^{e,k-1}\}$ corresponding to the aggregate income path of the level-$k-1$ equilibrium in a similar manner. Thus, $\hat{Y}_t^k$ solves the fixed point equation

$$\hat{Y}_t^k = C^*(\{\hat{R}_{t+s}\}, \hat{Y}_t^k, \{\hat{Y}_{t+1+s}^{k-1}\}).$$

We collect these in statements in a formal definition.

Definition (Level-k equilibrium)**.** Given an initial REE $\{R_t, Y_t\}$ and a new interest rate path $\{\hat{R}_t\}$, the level-k equilibrium $\{R_t, \hat{Y}_t^k\}$ is defined by a recursion indexed by $k \geq 0$ with initial condition $\{\hat{Y}_t^0\} = \{Y_t\}$, and such that $\{\hat{R}_t, \hat{Y}_t^k\}$ is a temporary equilibrium given beliefs $\{\hat{Y}_t^{e,k}\} = \{\hat{Y}_t^{k-1}\}$.

In the definitions of temporary and level-k equilibria, we include the actual present aggregate income, instead of some expectation over current aggregate income. This implies that markets clear in the present period and that basic macroeconomic identities hold. This impact of current aggregate income, however, will vanish in some cases in continuous time.

Decomposing equilibrium changes: PE and GE. Start at some baseline REE $\{R_t, Y_t\}$ and consider an new announced interest rate path $\{\hat{R}_t\}$ at $t = 0$.

Under rational expectations, the new equilibrium $\{\hat{R}_t, \hat{Y}_t\}$ is an REE. We can decompose the change in aggregate income

$$\Delta Y_t = \hat{Y}_t - Y_t$$

as

$$\Delta Y_t = \Delta Y_t^{PE} + \Delta Y_t^{GE},$$

where

$$\begin{aligned}
\Delta Y_t^{PE} &= C^*(\{\hat{R}_{t+s}\}, Y_t, \{Y_{t+1+s}\}) - C^*(\{R_{t+s}\}, Y_t, \{Y_{t+1+s}\}), \\
\Delta Y_t^{GE} &= C^*(\{\hat{R}_{t+s}\}, \hat{Y}_t, \{\hat{Y}_{t+1+s}\}) - C^*(\{\hat{R}_{t+s}\}, Y_t, \{Y_{t+1+s}\}).
\end{aligned}$$

The term ΔY_t^{PE} can be interpreted as a partial equilibrium effect considering the change in the interest rate only, holding constant current and future income. The term ΔY_t^{GE} captures the general equilibrium effects from changing current and future expected income, holding fixed the interest rates (at their new level).

Under level-k thinking, we denote the change in aggregate income by

$$\Delta Y_t^k = \hat{Y}_t^k - Y_t.$$

Note that we have

$$\Delta Y_t^k = \Delta Y_t^{PE} + \Delta Y_t^{k,GE},$$

with

$$\Delta Y_t^{k,GE} = C^*(\{\hat{R}_{t+s}\}, \hat{Y}_t^k, \{\hat{Y}_{t+1+s}^{k-1}\}) - C^*(\{\hat{R}_{t+s}\}, Y_t, \{Y_{t+1+s}\}).$$

In particular, since $\{\hat{Y}_t^0\} = \{Y_t\}$, the only reason why $\Delta Y_t^{1,GE} = C^*(\{\hat{R}_{t+s}\}, \hat{Y}_t^1, \{Y_{t+1+s}\}) - C^*(\{\hat{R}_{t+s}\}, Y_t, \{Y_{t+1+s}\})$ is not zero is due to the effect of the adjustment of current income $\hat{Y}_t^1$. As we shall see, this difference vanishes in some cases in continuous time. In these cases, level-1 thinking coincides exactly with the partial equilibrium effect.

Effects of monetary policy at different horizons. To summarize the effects of monetary policy at different horizons, we define the elasticities of output at date t to an interest rate change at date τ as follows.

We consider an initial REE $\{R_t, Y_t\}$ which for simplicity we assume is a steady state with $R_t = R$ and $Y_t = Y$ for all $t \geq 0$. We consider a change $\{\hat{R}_t\}$ in the path for the interest rate ΔR_τ at date τ so that $\hat{R}_\tau = R + \Delta R_\tau$ and $\hat{R}_t = R_t$ for $t \neq \tau$. The rational expectation elasticity is defined as

$$\epsilon_{t,\tau} = \lim_{\Delta R_\tau \to 0} -\frac{R_\tau}{Y_t}\frac{\Delta Y_t}{\Delta R_\tau},$$

and can be decomposed as

$$\epsilon_{t,\tau} = \epsilon_{t,\tau}^{PE} + \epsilon_{t,\tau}^{GE},$$

where

$$\epsilon_{t,\tau}^{PE} = \lim_{\Delta R_\tau \to 0} -\frac{R_\tau}{Y_t}\frac{\Delta Y_t^{PE}}{\Delta R_\tau},$$
$$\epsilon_{t,\tau}^{GE} = \lim_{\Delta R_\tau \to 0} -\frac{R_\tau}{Y_t}\frac{\Delta Y_t^{GE}}{\Delta R_\tau}.$$

Similarly, the level-k elasticity is defined as

$$\epsilon_{t,\tau}^{k} = \lim_{\Delta R_\tau \to 0} -\frac{R_\tau}{Y_t}\frac{\Delta Y_t^{k}}{\Delta R_\tau}.$$

Remark. An immediate consequence of the fact that the aggregate model is purely forward looking is that all these elasticities are zero whenever $t > \tau$. We will therefore focus on the case where $t \leq \tau$.

2.2 Extended Model with a State Variable

The previous analysis is sufficient for the simplest cases, such as the representative agent and the perpetual youth overlapping generations models. Aggregate consumption is purely forward looking in these cases. However, in an incomplete-markets Aiyagari-Bewley economy, the distribution of wealth induces a backward looking component. Likewise, in an open economy model, the stock of a countries assets or wealth, accumulated in the past, would also affect consumption. To incorporate these effects we now extend the analysis to include an aggregate state variable.

Suppose aggregate consumption is given by

$$C_t = C^*(\{R_{t+s}\}, Y_t, \{Y^e_{t+1+s}\}, \Psi_t), \tag{4}$$

where the state variable Ψ_t is potentially of a large dimension and evolves according to some equilibrium law of motion

$$\Psi_{t+1} = M(\{R_{t+s}\}, Y_t, \{Y^e_{t+1+s}\}, \Psi_t). \tag{5}$$

The initial state Ψ_0 is taken as given. In incomplete market economies, Ψ_t may capture the distribution of wealth and M captures the evolution of the wealth distribution. The important point is that the aggregate consumption is no longer purely forward looking.

We can easily extend all our definitions. A temporary equilibrium given beliefs $\{Y^e_t\}$ is a set of sequences $\{R_t, Y_t, \Psi_t\}$ satisfying (2), (4), and (5) for all $t \geq 0$. An REE is a set of sequences $\{R_t, Y_t, Y^e_t, \Psi_t\}$ such that $\{R_t, Y_t, \Psi_t\}$ is a temporary equilibrium given beliefs $\{Y^e_t\}$ and which satisfies perfect foresight (3) for all $t = 0, 1, \ldots$ Given a baseline REE and an new announced interest rate path $\{\hat{R}_t\}$ at $t = 0$, level-k equilibria $\{\hat{R}_t, \hat{Y}^k_t, \hat{\Psi}^k_t\}$ are defined by a recursion indexed by $k \geq 0$ with initial condition $\{\hat{Y}^0_t\} = \{Y_t\}$, and such that $\{\hat{R}_t, \hat{Y}^k_t, \hat{\Psi}^k_t\}$ is a temporary equilibrium given beliefs $\{\hat{Y}^{e,k}_t\} = \{\hat{Y}^{k-1}_t\}$. Armed with these definitions, it is straightforward to extend the definitions of the elasticities $\epsilon_{t,\tau}$, $\epsilon^{PE}_{t,\tau}$, $\epsilon^{GE}_{t,\tau}$, and $\epsilon^k_{t,\tau}$.

Remark. Since the model is no longer necessarily purely forward looking, it is no longer true that all these elasticities are zero for $t > \tau$.

3 The Representative Agent Model

In this section, we consider the particular case of a representative agent model with per period utility function U in a Lucas tree economy with a unit supply of Lucas trees with time-t value V_t capitalizing a stream δY_t of dividends and with non-financial (labor) income given by $(1-\delta)Y_t$. The representative agent can invest in Lucas trees and also borrow and lend in short-term risk-free bonds with the sequence of interest rates $\{R_t\}$. At every point in time t, the agents has beliefs $\{Y^e_{t+1+s}, V^e_{t+1+s}\}$ about future aggregate income and Lucas tree values.

In Section 3.1, we show how to derive the reduced-form aggregate consumption function from the consumption policy function of an individual problem using the asset market clearing condition for a general utility function. We then leverage all the definitions of

Section 2.1: temporary equilibria, rational expectations equilibria, level-k equilibria, and the corresponding interest rate elasticities. In Section 3.2, we specialize the model to the case of an isoelastic utility function and derive analytical results.

3.1 The General Representative Agent Model

In this section, we assume a general utility function U.

Individual problem. Consider sequences $\{R_t, Y_t, Y_t^e, V_t, V_t^e\}$. An individual takes these sequences as given. At every point in time t, current consumption c_t, current bond and Lucas tree holdings b_t and x_t are determined as a function of past bond and Lucas tree holdings b_{t-1} and x_{t-1} via the individual policy functions

$$
\begin{aligned}
c_t &= c^*(b_{t-1}, x_{t-1}; \{R_{t+s}\}, Y_t, \{Y^e_{t+1+s}\}, V_t, \{V^e_{t+1+s}\}), \\
b_t &= b^*(b_{t-1}, x_{t-1}; \{R_{t+s}\}, Y_t, \{Y^e_{t+1+s}\}, V_t, \{V^e_{t+1+s}\}), \\
x_t &= x^*(b_{t-1}, x_{t-1}; \{R_{t+s}\}, Y_t, \{Y^e_{t+1+s}\}, V_t, \{V^e_{t+1+s}\}).
\end{aligned}
$$

This defines a recursion over t, which together with the initial conditions $b_{-1} = 0$ and $x_{-1} = 1$, entirely determines individual sequences $\{c_t, b_t, x_t\}$.

The individual policy functions at time t are derived from the following individual problem at time t, given b_{t-1} and x_{t-1}:

$$\max_{\{\tilde{c}_{t+s}, \tilde{b}_{t+s}, \tilde{x}_{t+s}\}} \sum_{s=0}^{\infty} \beta^s U(\tilde{c}_{t+s})$$

subject to the current actual budget constraint

$$\tilde{c}_t = (1-\delta)Y_t + x_{t-1}V_t + b_{t-1}R_{t-1} - \tilde{x}_t V_t - \tilde{b}_t,$$

and future expected budget constraints

$$\tilde{c}_{t+1+s} = (1-\delta)Y^e_{t+1+s} + \tilde{x}_{t+s}V^e_{t+1+s} + \tilde{b}_{t+s}R_{t+s} - \tilde{x}_{t+1+s}V^e_{t+1+s} - \tilde{b}_{t+1+s} \quad \forall s \geq 0.$$

We define $c^*(b_{t-1}, x_{t-1}; \{R_{t+s}\}, Y_t, \{Y^e_{t+1+s}\}, V_t, \{V^e_{t+1+s}\})$, to be the value of $\tilde{c}_t$ at the optimum. Similarly, $b^*(b_{t-1}, x_{t-1}; \{R_{t+s}\}, Y_t, \{Y^e_{t+1+s}\}, V_t, \{V^e_{t+1+s}\})$ is the value of $\tilde{b}_t$ at the optimum, and $x^*(b_{t-1}, x_{t-1}; \{R_{t+s}\}, Y_t, \{Y^e_{t+1+s}\}, V_t, \{V^e_{t+1+s}\})$ is the value of $\tilde{x}_t$ at the optimum. These values satisfy the current actual budget constraint.[5]

[5] Importantly, note that at the optimum $\{\tilde{c}_{t+1+s}, \tilde{b}_{t+1+s}, \tilde{x}_{t+1+s}\}$ is in general different from

We now simplify these steps by imposing the following necessary no arbitrage conditions for the individual problems to have a solution:

$$V_t = \delta Y_t + \frac{V^e_{t+1}}{R_t} \quad \forall t \geq 0,$$
$$V^e_t = \delta Y^e_t + \sum_{s=0}^{\infty} \frac{\delta Y^e_{t+1+s}}{\Pi^s_{u=0} R_{t+u}} \quad \forall t \geq 0. \tag{6}$$

Given no arbitrage, an individual agents is indifferent between bonds and Lucas trees, and the composition of his portfolio is indeterminate. Accordingly, we define a new variable $a_t = b_{t-1}R_{t-1} + x_{t-1}(\delta Y_t + V_t)$ denoting financial wealth at time t.

We can then simplify the individual problem at time t:

$$\max_{\{\tilde{c}_t, \tilde{a}_{t+1+s}\}} \sum_{s=0}^{\infty} \beta^s U(\tilde{c}_{t+s})$$

subject to the current actual budget constraint

$$\tilde{c}_t = (1-\delta)Y_t + a_t - \frac{\tilde{a}_{t+1}}{R_t},$$

and future expected budget constraints

$$\tilde{c}_{t+1+s} = (1-\delta)Y^e_{t+1+s} + \tilde{a}_{t+1+s} - \frac{\tilde{a}_{t+2+s}}{R_{t+1+s}} \quad \forall s \geq 0,$$

We denote by $c^*(a_t; \{R_{t+s}\}, Y_t, \{Y^e_{t+1+s}\})$ and $a^*(a_t; \{R_{t+s}\}, Y_t, \{Y^e_{t+1+s}\})$ the individual policy functions of the individual problem at time t. They are given by the values of time-t consumption $\tilde{c}_t$ and time-$t+1$ assets $\tilde{a}_{t+1}$ at the individual optimum. These values satisfy the current actual time-t budget constraint. Note that V_t and $\{V^e_{t+1+s}\}$ are no longer arguments of these policy functions, a very convenient simplification.

At every point in time t, current consumption c_t and financial wealth a_{t+1} are determined as a function of past financial wealth a_t via the individual policy functions

$$c_t = c^*(a_t; \{R_{t+s}\}, Y_t, \{Y^e_{t+1+s}\}),$$
$$a_{t+1} = a^*(a_t; \{R_{t+s}\}, Y_t, \{Y^e_{t+1+s}\}).$$

This defines a recursion over t, which together with the initial conditions $a_0 = V_0$ entirely

$\{c_{t+1+s}, b_{t+1+s}, x_{t+1+s}\}$. This is a symptom of time inconsistency when beliefs deviate from rational expectations.

determines individual sequences $\{c_t, a_t\}$.

Reduced-form aggregate consumption function. The reduced-form aggregate consumption is obtained from the individual consumption function $C(\{R_{t+s}\}, Y_t, \{Y^e_{t+1+s}\}) = c(a_t; \{R_{t+s}\}, Y_t, \{Y^e_{t+1+s}\})$ and imposing the asset market clearing condition $a_t = V_t$, where V_t is given by the no-arbitrage condition (6). This yields

$$C(\{R_{t+s}\}, Y_t, \{Y^e_{t+1+s}\}) = c^*(\delta Y_t + \sum_{s=1}^{\infty} \frac{\delta Y^e_{t+1}}{\Pi_{u=0}^{s} R_{t+u}}; \{R_{t+s}\}, Y_t, \{Y^e_{t+1+s}\}).$$

We can then use this reduced-form aggregate consumption function to go through all the definitions given in Section 2: temporary equilibria, rational expectations equilibria, level-k equilibria, and the corresponding interest rate elasticities.

3.2 Isoelastic Utility Function

In this section, we specialize the model to the case of an isoelastic utility function with intertemporal elasticity of substitution σ:

$$U(c) = \begin{cases} \frac{c^{1-\frac{1}{\sigma}}-1}{1-\frac{1}{\sigma}} & \text{if} \quad \sigma \neq 1, \\ \log(c) & \text{if} \quad \sigma = 1. \end{cases}$$

It is then easy to see that the individual consumption function is

$$c^*(a_t; \{R_{t+s}\}, Y_t, \{Y^e_{t+1+s}\}) = \frac{a_t + (1-\delta)Y_t + \sum_{s=0}^{\infty} \frac{(1-\delta)Y^e_{t+1+s}}{\Pi_{u=0}^{s} R_{t+u}}}{1 + \sum_{s=0}^{\infty} \frac{\beta^{\sigma(1+s)}}{\Pi_{u=0}^{s} R_{t+u}{}^{1-\sigma}}},$$

so that the aggregate reduced-form consumption function is

$$C(\{R_{t+s}\}, Y_t, \{Y^e_{t+1+s}\}) = \frac{Y_t + \sum_{s=0}^{\infty} \frac{Y^e_{t+1+s}}{\Pi_{u=0}^{s} R_{t+u}}}{1 + \sum_{s=0}^{\infty} \frac{\beta^{\sigma(1+s)}}{\Pi_{u=0}^{s} R_{t+u}{}^{1-\sigma}}}.$$

Equilibrium characterization. For concreteness, we briefly characterize the various equilibria in the context of this particular model. Given beliefs $\{Y^e_t\}$, and given the path for interest rates $\{R_t\}$, $\{R_t, Y_t\}$ is a temporary equilibrium if and only if the path for aggregate income $\{Y_t\}$ is given by

$$Y_t = \frac{\sum_{s=0}^{\infty} \frac{Y^e_{t+1+s}}{\Pi_{u=0}^{s} R_{t+u}}}{\sum_{s=0}^{\infty} \frac{\beta^{\sigma(1+s)}}{\Pi_{u=0}^{s} R_{t+u}{}^{1-\sigma}}} \quad \forall t \geq 0.$$

Similarly, given the path for interest rates $\{R_t\}$, $\{R_t, Y_t\}$ is an REE if and only if the path for aggregate income $\{Y_t\}$ satisfies the fixed point

$$Y_t = \frac{\sum_{s=0}^{\infty} \frac{Y_{t+1+s}}{\Pi_{u=0}^{s} R_{t+u}}}{\sum_{s=0}^{\infty} \frac{\beta^{\sigma(1+s)}}{\Pi_{u=0}^{s} R_{t+u}{}^{1-\sigma}}} \quad \forall t \geq 0.$$

Finally given an initial REE $\{R_t, Y_t\}$ and a new interest rate path $\{\hat{R}_t\}$, the level-k equilibria $\{\hat{R}_t, \hat{Y}^k_t\}$ satisfy the following recursion over $k \geq 0$:

$$\hat{Y}^k_t = \frac{\sum_{s=0}^{\infty} \frac{\hat{Y}^{k-1}_{t+1+s}}{\Pi_{u=0}^{s} \hat{R}_{t+u}}}{\sum_{s=0}^{\infty} \frac{\beta^{\sigma(1+s)}}{\Pi_{u=0}^{s} \hat{R}_{t+u}{}^{1-\sigma}}} \quad \forall t \geq 0,$$

with the initialization that $\hat{Y}^0_t = Y_t$ for all $t \geq 0$.

We now turn to the computation of the different interest rate elasticities of output around a steady state REE $\{R_t, Y_t\}$ with $R_t = R = \beta^{-1} > 1$ and $Y_t = Y > 0$ for all $t \geq 0$.

Monetary policy at different horizons under RE. We start with the RE case, where we use the selection that $\lim_{t\to\infty} Y_t = Y$ as we perform the comparative statics underlying the computation of the interest rate elasticities of output.

Proposition 1 (Representative agent, isoelastic utility, RE). *Consider the representative agent model with isoelastic utility and rational expectations. For $t > \tau$ the interest rate elasticities of output are zero $\epsilon_{t,\tau} = 0$. For $t \leq \tau$, they depend only on the horizon $\tau - t$ are given by*

$$\epsilon_{t,\tau} = \sigma.$$

They can be decomposed as $\epsilon_{t,\tau} = \epsilon^{PE}_{t,\tau} + \epsilon^{GE}_{t,\tau}$ into PE and GE elasticities $\epsilon^{PE}_{t,\tau}$ and $\epsilon^{GE}_{t,\tau}$. For $t > \tau$ these elasticities are zero $\epsilon^{PE}_{t,\tau} = \epsilon^{GE}_{t,\tau} = 0$. For $t \leq \tau$, they depend only on the horizon $\tau - t$ and are given by

$$\epsilon^{PE}_{t,\tau} = \sigma \frac{1}{R^{\tau-t+1}} \quad \text{and} \quad \epsilon^{GE}_{t,\tau} = \sigma\left(1 - \frac{1}{R^{\tau-t+1}}\right).$$

Because the aggregate model is purely forward looking, the interest rate elasticity $\epsilon_{t,\tau}$

of output at date t to interest rate changes at date τ is zero for $t > \tau$. From now on we focus on $t \leq \tau$ and we call $\tau - t$ the horizon of monetary policy.

The total interest rate elasticity of output is equal to the intertemporal elasticity of substitution $\epsilon_{t,\tau} = \sigma$ independently of the horizon $\tau - t$. This lack of horizon effect is a version of the "forward guidance puzzle", which refers to the extreme effectiveness of forward guidance (interest rate changes in the future) in standard New-Keynesian models compared to its apparently more limited effectiveness in the data.

To understand this result, it it useful to go back to the decomposition into a PE effect and a GE effect. The lack of *horizon* effect

$$\frac{\partial \epsilon_{t,\tau}}{\partial \tau} = 0$$

can be understood as follows, where, slightly abusing notation, we write $\frac{\partial \epsilon_{t,\tau}}{\partial \tau}$ for $\epsilon_{t,\tau+1} - \epsilon_{t,\tau}$. The PE effect does feature a horizon effect so that $\epsilon^{PE}_{t,\tau}$ is decreasing with the horizon $\tau - t$ with

$$\frac{\partial \epsilon^{PE}_{t,\tau}}{\partial \tau} = -\log(R)\epsilon^{PE}_{t,\tau} < 0.$$

This is because for a given path of output, a cut in interest rates is more discounted, and hence leads to a smaller partial equilibrium consumption increase, the further into the future the interest rate cut takes place. But the GE effect features an exactly offsetting anti-horizon effect so that $\epsilon^{GE}_{t,\tau}$ increases with the horizon $\tau - t$ with

$$\frac{\partial \epsilon^{GE}_{t,\tau}}{\partial \tau} = -\frac{\partial \epsilon^{PE}_{t,\tau}}{\partial \tau} > 0.$$

This is because in general equilibrium, output increases for a longer time (until the horizon of the interest rate cut), leading to a higher increase in human and financial wealth, the further into the future the interest rate cut takes place, and hence leads to a larger consumption increase. As a result, the relative importance of the GE effect increases with the horizon, and that of the PE effect correspondingly decreases with the horizon, but the two effects always sum up to a constant total effect σ.

Monetary policy at different horizons under level-k. We now turn to the level-k case. We start by defining the function

$$\mathcal{E}^k(R-1, \tau - t) = \sum_{m=0}^{k} (R-1)^m \sum_{s_0=0}^{\tau-t-1} \sum_{s_1=0}^{\tau-t-1-s_0} \cdots \sum_{s_{m-2}=0}^{\tau-t-1-s_{m-3}} 1.$$

The function $\mathcal{E}^k$ is increasing in k with $\mathcal{E}^1(R-1, \tau - t, 1) = 1$ and $\lim_{k\to\infty} \mathcal{E}^k(R-1, \tau - t) = R^{\tau-t}$.[6]

Proposition 2 (Representative agent, level-k)**.** *Consider the representative agent model with isoelastic utility and level-k thinking. For $t > \tau$ the interest rate elasticities of output are zero $\epsilon^k_{t,\tau} = 0$. For $t \leq \tau$, they depend only on the horizon $\tau - t$ and are given by*

$$\epsilon^k_{t,\tau} = \sigma \frac{\mathcal{E}^k(R-1, \tau - t)}{R^{\tau-t}}.$$

As above, we focus on the interesting case $t \leq \tau$. Recall that the PE effect is always the same under rational expectations of level-k thinking at $\epsilon^{PE}_{t,\tau}$. The level-1 elasticity is always higher than the PE effect by a factor of R since

$$\epsilon^1_{t,\tau} = \sigma \frac{1}{R^{\tau-t}} = R\epsilon^{PE}_{t,\tau} > \epsilon^{PE}_{t,\tau},$$

but as we shall see below, the difference $\epsilon^{1,GE}_{t,\tau} = \epsilon^1_{t,\tau} - \epsilon^{PE}_{t,\tau}$ vanishes in the continuous time limit where time periods become infinitesimal so that the per-period interest rate R shrinks to 1. The interest rate elasticity of output with level-k thinking is lower than under rational expectations

$$\epsilon^k_{t,\tau} < \epsilon_{t,\tau},$$

but increases with the level of thought k

$$\frac{\partial \epsilon^k_{t,\tau}}{\partial k} > 0,$$

and converges to its rational expectations counterpart in the limit $k \to \infty$

$$\lim_{k\to\infty} \epsilon^k_{t,\tau} = \epsilon_{t,\tau},$$

where, slightly abusing notation, we write $\frac{\partial \epsilon^k_{t,\tau}}{\partial k}$ for $\epsilon^{k+1}_{t,\tau} - \epsilon^{k+1}_{t,\tau}$. The *mitigation* effect $\epsilon^k_{t,\tau} < \epsilon_{t,\tau}$ is entirely due to a mitigation of the GE effect $\epsilon^{k,GE}_{t,\tau} < \epsilon^{GE}_{t,\tau}$. Similarly, the

[6] It is also useful to compute a few other examples explicitly. We have

$$\mathcal{E}^1(R-1, \tau - t, 1) = 1,$$
$$\mathcal{E}^2(R-1, \tau - t, 2) = 1 + (R-1)(\tau - t),$$
$$\mathcal{E}^3(R-1, \tau - t, 3) = 1 + (R-1)(\tau - t) + \frac{(R-1)^2(\tau - t - 1)(\tau - t)}{2}.$$

monotonically increasing *convergence* $\lim_{k\to\infty} \epsilon^k_{t,\tau} = \epsilon_{t,\tau}$ is entirely due to the convergence of the monotonically increasing convergence of the GE effect $\lim_{k\to\infty} \epsilon^{k,GE}_{t,\tau} = \epsilon^{GE}_{t,\tau}$. The rational expectations case can therefore be seen as a limit case of level-k thinking as the number of rounds k goes to ∞.

In addition, for any $k > 0$, in contrast to the rational expectations case, there is now a *horizon* effect of monetary policy

$$\frac{\partial \epsilon^k_{t,\tau}}{\partial \tau} < 0,$$

so that the effects of monetary policy decrease with its horizon. This horizon effect disappears in the rational expectations limit $k \to \infty$.

However, note that the mitigation and horizon effects are rather *weak*. To see this focus on the case $k = 1$. Then $\epsilon^1_{t,\tau} = \sigma \frac{1}{R^{\tau-t}}$ and so

$$\frac{\partial \epsilon^1_{t,\tau}}{\partial \tau} = -\log(R)\epsilon^1_{t,\tau}.$$

Hence $\epsilon^1_{t,\tau} = \epsilon_{t,\tau}$ when the interest rate change is contemporaneous $\tau - t = 0$ and then $\epsilon^1_{t,\tau}$ decreases with the horizon $\tau - t$ at the exponential rate $\log(R)$ while $\epsilon_{t,\tau} = \sigma$ stays constant. We call $\log(R)$ the strength of the horizon effect. If the annual interest rate is 5%, the effects of monetary policy decrease at rate 5% per year with a half life of 14 years; if the annual interest rate is 1%, the effects of monetary policy decrease at rate 1% per year with a half life of 69 years.

There is a simple intuition for all these results in terms of the decomposition of the effects of monetary policy into PE and GE effects. The PE effect features mitigation—the effect of interest rate changes is lower than the full effect under rational expectations because the latter is the sum of the GE and the PE effect. It also features horizon—for a fixed path of output, interest rate changes affect partial equilibrium consumption less, the further in the future they are. These effects are weak for reasonable values of R. As we shall see below, this last conclusion can be overturned in models with heterogenous agents and incomplete markets.

Under rational expectations, the GE effect eliminates the mitigation feature by adding to the PE effect, and eliminates the horizon effect because the GE effect features an anti-horizon effect. At round $k = 1$, monetary policy almost (and exactly in the continuous time limit) coincides with the PE effect and features weak mitigation and weak horizon. In the rational expectations limit $k \to \infty$, the mitigation and horizon effects disappear. Intermediate values of k interpolate smoothly and monotonically between these two extremes.

Remark. It is interesting to note that the various interest rate elasticities of output are all independent of the amount of outside liquidity δ. This is because human and financial wealth play very similar roles in this representative agent model. As we shall see shortly, this irrelevance result can be overturned (especially with level-k thinking) in heterogenous agents models with incomplete markets.

3.3 Continuous-Time Limit

We now explain how the results can be adapted in continuous time. This can be done either directly by setting up the model in continuous time, or by taking the continuous time limit of the discrete time model. In Section 4, we follow the former approach. In this section instead, we follow the latter.

The continuous-time limit involves considering a sequence of economies indexed by $n \geq 0$, where the calendar length λ_n of a period decreases with n. For example, we can take $\lambda_n = \frac{1}{n}$. We keep the discount factor constant per unit of calendar time as we increase n requires by imposing that the discount factor per period equal $\beta_n = e^{\rho \lambda_n}$ for some instantaneous discount rate ρ. The steady-state interest rate is then constant per unit of calendar time as we increase n, but the interest rate per period is $R_n = e^{r\lambda_n}$ for the instantaneous interest rate $r = \rho$. This naturally implies that $\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} R_n = 1$. Note that a given calendar date t corresponds to a different period number $t_n(t) = \frac{t}{\lambda_n}$ for different values of n.

We can then apply our definitions from the previous sections for every value of n and take the limit as n goes to ∞. For fixed calendar date t and τ, we can compute the limits of $\epsilon_{t_n(t),t_n(\tau)}$, $\epsilon^{PE}_{t_n(t),t_n(\tau)}$, $\epsilon^{GE}_{t_n(t),t_n(\tau)}$, $\epsilon^{k}_{t_n(t),t_n(\tau)}$, and $\epsilon^{k,GE}_{t_n(t),t_n(\tau)}$ when n goes to ∞. We denote these limits by $\epsilon_{t,\tau}$, $\epsilon^{PE}_{t,\tau}$, $\epsilon^{GE}_{t,\tau}$, $\epsilon^{k}_{t,\tau}$, and $\epsilon^{k,GE}_{t,\tau}$. They represent the elasticity of output at date t to a localized cumulated interest rate change Δr_τ at date τ, by which we mean a change in the interest rate path $\{\hat{r}_t\}$ given by $\hat{r}_t = r + \Delta r_\tau \delta_\tau(t)$ where δ_τ is the Dirac function so that $\int_0^t (\hat{r}_u - r)du = 0$ for $t < \tau$ and $\int_0^t (\hat{r}_u - r)du = \Delta r_\tau$ for $t > \tau$.

We also define the continuous-time analogue $\mathcal{E}^k_{ct}(r(\tau - t))$ of $\mathcal{E}^k(R-1, \tau - t)$:

$$\mathcal{E}^k_{ct}(r(\tau - t)) = \sum_{m=0}^{k} \frac{[r(\tau - t)]^m}{m!},$$

where $\mathcal{E}^k_{ct}(r(\tau - t))$ is increasing in k with $\mathcal{E}^1_{ct}(r(\tau - t)) = 1$ and $\lim_{k\to\infty} \mathcal{E}^k_{ct}(r(\tau - t)) = e^{r(\tau - t)}$.

Proposition 3 (Representative agent, continuous time)**.** *Consider the representative agent* *model with isoelastic utility and either rational expectations or level-k thinking. For $t > \tau$ the interest rate elasticities of output are zero $\epsilon_{t,\tau} = \epsilon^k_{t,\tau} = 0$. For $t \leq \tau$, they depend only on the horizon $\tau - t$ and are given by*

$$\epsilon_{t,\tau} = \sigma, \quad \epsilon^{PE}_{t,\tau} = \sigma e^{-r(\tau-t)}, \quad \epsilon^{GE}_{t,\tau} = \sigma[1 - e^{-r(\tau-t)}],$$

$$\epsilon^k_{t,\tau} = \sigma e^{-r(\tau-t)} \mathcal{E}^k_{ct}(r(\tau - t)).$$

All of our other results go through and the intuitions are identical. In particular, level-k thinking features (weak) mitigation $\epsilon^k_{t,\tau} < \epsilon_{t,\tau}$, and monotonic convergence with $\frac{\partial \epsilon^k_{t,\tau}}{\partial k} > 0$ and $\lim_{k\to\infty} \epsilon^k_{t,\tau} = 1$. Compared to the discrete-time case, a useful simplification occurs for $k = 1$ since now have

$$\epsilon^1_{t,\tau} = \epsilon^{PE}_{t,\tau} = \sigma e^{-r(\tau-t)},$$

so that level-1 now coincides exactly (and not just approximately) with the PE effect. As a result, the (weak) horizon effect is now given by

$$\frac{\partial \epsilon^1_{t,\tau}}{\partial \tau} = -r\epsilon^1_{t,\tau},$$

so that its strength is simply r. This is because in continuous time, the impact of current income on current consumption vanishes, since it becomes a vanishing fraction of permanent income.

4 The Perpetual-Youth Model of Borrowing Constraints

In this section we introduce a standard overlapping generations model of the "perpetual youth" variety a la Blanchard-Yaari. As is well known, overlapping generation models can be reinterpreted as models with heterogenous agents subject to borrowing constraints (e.g. Woodford, 1990, Kocherlakota 1992). The death event under the finite lifetime interpretation represents a binding borrowing constraint in the other interpretation. The important common property is that horizons are shortened in that consumption is only smoothed over fewer periods.

We offer an explicit interpretation along these lines. The perpetual youth setup with homothetic preferences and annuities allows us to neatly isolate the impact of occasionally binding borrowing constraints while getting rid of precautionary savings. In Section 5, we will consider a standard Aiyagari-Bewley economy with both occasionally binding borrowing constraints and precautionary savings.

We set up the model directly in continuous time for tractability. The economy is populated by infinitely-lived agents randomly hit by idiosyncratic discount factor shocks that make borrowing constraint bind according to a Poisson process. There is unit mass of ex-ante identical atomistic agents indexed by i which is uniformly distributed over $[0, 1]$.

We assume that per-period utility U is isoelastic with a unitary intertemporal elasticity of substitution $\sigma = 1$ which simplifies the analysis. We refer the reader to the appendix for the case $\sigma \neq 1$.

We allow for positive outside liquidity in the form of Lucas trees in unit-supply with time-t value V_t capitalizing a stream δY_t of dividends and with non-financial (labor) income given by $(1 - \delta)Y_t$, the ownership of which at date 0 is uniformly distributed across agents. At every date, non-financial income is distributed uniformly across the population.

Agents can invest in Lucas trees and also borrow and lend in short-term risk-free bonds with the sequence of instantaneous interest rates $\{r_t\}$ subject to their borrowing constraints, and can also purchase actuarially fair annuities.

The presence of annuity markets drastically simplifies the analysis by neutralizing precautionary savings. Together with homothetic preferences, it implies that the model aggregates linearly. Therefore, no extra state variable is required to characterize the aggregate equilibrium.

Individual problem. We first describe the individual problem. We proceed as in Section 3.1 to formulate the individual problem given the aggregate paths $\{Y_t, Y_t^e, r_t\}$ directly in terms of total financial wealth a_t^i as long as Lucas trees satisfy the no-arbitrage conditions

$$V_t = \int_0^\infty \delta Y_{t+s} e^{-\int_0^s r_{t+u} du} ds. \tag{7}$$

Agents are hit by idiosyncratic Poisson shocks with intensity λ. The life of an agent i is divided into "periods" by the successive realizations n of his idiosyncratic Poisson process indexed by the stopping times τ_n^i with $\tau_0^i = 0$ by convention. The agent has a low discount factor $\beta < 1$ between the different periods, and an instantaneous discount rate ρ within each period. Importantly, the agent cannot borrow against his future non-financial or human wealth accruing in any future "period". In other words, for $\tau_n^i \leq t < \tau_{n+1}^i$, agent i cannot borrow against any future non-financial income or human wealth accruing after τ_{n+1}^i. We assume that the discount factor $\beta < 1$ is sufficiently low that agents are up against their borrowing constraints between two "periods", so that in equilibrium, agents always choose not to bring in any financial wealth from one "period" to the next and hence $a^i_{\tau^i_{n+1}} = 0$ for all $n \geq 0$ and $i \in [0,1]$, where a^i_t denotes the financial wealth of agent i at time t. The parameter λ can then be thought of as indexing the frequency of binding borrowing constraints.

The problem on an at date t with financial wealth a^i_t and who is in "period" n_t is therefore

$$\max_{\{\tilde{c}^i_{t+s}, \tilde{a}^i_{t+s}\}} \mathbb{E}_t \sum_{n=0}^{\infty} \beta^n \int_{\tau^i_{n_t+n}}^{\tau^i_{n_t+n+1}} \log(\tilde{c}^i_{t+s}) e^{-\rho s} ds,$$

subject to the future expected budget constraints

$$\frac{d\tilde{a}^i_{t+s}}{ds} = (r_{t+s} + \lambda)\tilde{a}^i_{t+s} + Y^e_{t+s} - \tilde{c}^i_{t+s} \quad \text{for} \quad \tau^i_{n_t+n} \leq t+s < \tau^i_{n_t+n+1},$$

to the initial condition

$$\tilde{a}^i_t = a^i_t,$$

and the borrowing constraints

$$\tilde{a}^i_{\tau_{n_t+n+1}} = 0 \quad \forall n \geq 0.$$

The policy function for consumption at date t is the individual consumption function and is given by

$$c^*(a^i_t; \{r_{t+s}\}, \{Y^e_{t+s}\}) = (\rho + \lambda)[a^i_t + \int_0^{\infty} (1-\delta) Y^e_{t+s} e^{-\int_0^s (r_{t+u}+\lambda) du} ds].$$

Note that this policy function is independent of the "period" n because the idiosyncratic Poisson process is memoryless. It depends only on expected future income $\{Y^e_{t+s}\}$ but not on current income Y_t, because of the continuous time assumption.

The law of motion for a^i_t is given by the actual (as opposed to expected) budget constraints

$$\frac{da^i_t}{dt} = (r_t + \lambda)a^i_t + Y_t - c^*(a^i_t; \{r_{t+s}\}, \{Y^e_{t+s}\}) \quad \text{for} \quad \tau^i_{n_t+n} \leq t+s < \tau^i_{n_t+n+1},$$

the initial condition

$$a^i_0 = V_t,$$

and the borrowing constraints

$$a^i_{\tau_n} = 0 \quad \forall n \geq 1.$$

Aggregate state variable. The model also features an aggregate state variable as in Section 2.2: the wealth distribution $\Psi_t = \{a_t^i\}$. which is the aggregate state variable for this model. The law of motion for Ψ_t is entirely determined by the laws of motion for individual financial wealth a_t^i. However as we shall see below, this aggregate state variable is not required to characterize the aggregate equilibrium.

Reduced-form aggregate consumption function. The reduced-form aggregate consumption function is obtained by aggregating over i the individual consumption function $C(\{r_{t+s}\}, \{Y_{t+s}^e\}) = \int_0^1 c^*(a_t^i; \{r_{t+s}\}, \{Y_{t+s}^e\})di$ and imposing the asset market clearing condition $\int a_t^i di = V_t$, where V_t is given by the no-arbitrage condition (7). This yields

$$C(\{r_{t+s}\}, \{Y_{t+s}^e\}) = (\rho + \lambda)[\int_0^\infty \delta Y_{t+s}^e e^{-\int_0^s r_{t+u}du} ds + \int_0^\infty (1-\delta) Y_{t+s}^e e^{-\int_0^s (r_{t+u}+\lambda)du} ds].$$

Just like the individual consumption function, and for the same reason, the reduced-form aggregate consumption function depends only on expected future income $\{Y_{t+s}^e\}$ but not on current income Y_t. More importantly, the aggregate consumption function is independent of the aggregate state variable $\Psi_t = \{a_t^i\}$.

Remarkably, the only difference in the reduced form the aggregate consumption function from the representative agent model analyzed in Sections 3.2-3.3 is that future expected aggregate non-financial income $(1-\delta)Y_{t+s}^e$ is discounted at rate $e^{-\int_0^s (r_{t+u}+\lambda)du}$ instead of $e^{-\int_0^s r_{t+u}du}$. Note however that future expected aggregate financial income δY_{t+s}^e, incorporated in the value of Lucas trees V_t, is still discounter at rate $e^{-\int_0^s r_{t+u}du}$. This is intuitive since borrowing constraints limit agents' ability to borrow against future non-financial income but does not prevent them from selling their assets when they are borrowing constraints.[7]The representative agent model can be obtained as the limit of this model when the frequency λ of binding borrowing constraints goes to zero.

Equilibrium characterization. For concreteness, we briefly characterize the various equilibria in the context of this particular model. Given beliefs $\{Y_t^e\}$, and given the path for interest rates $\{r_t\}$, $\{r_t, Y_t\}$ is a temporary equilibrium if and only if the path for aggregate

[7]Note that this requires financial assets to be liquid. Financial income (dividends) from partly illiquid assets should be discounted at a higher rate. For example, suppose that a fraction of trees can be sold while others cannot (or at a very large cost). Illiquid trees should then be treated like non-financial income. The financial income of illiquid trees should be discounted at rate $e^{-\int_0^s (r_{t+u}+\lambda)du}$ while that of liquid trees should be discounted at rate $e^{-\int_0^s r_{t+u}du}$. In essence, introducing illiquid trees is isomorphic to a reduction in δ.

income $\{Y_t\}$ is given by

$$Y_t = (\rho+\lambda)[\int_0^\infty \delta Y^e_{t+s} e^{-\int_0^s r_{t+u}du} ds + \int_0^\infty (1-\delta) Y^e_{t+s} e^{-\int_0^s (r_{t+u}+\lambda)du} ds] \quad \forall t \geq 0.$$

Similarly, given the path for interest rates $\{r_t\}$, $\{r_t, Y_t\}$ is an REE if and only if the path for aggregate income $\{Y_t\}$ satisfies the fixed point

$$Y_t = (\rho+\lambda)[\int_0^\infty \delta Y_{t+s} e^{-\int_0^s r_{t+u}du} ds + \int_0^\infty (1-\delta) Y_{t+s} e^{-\int_0^s (r_{t+u}+\lambda)du} ds] \quad \forall t \geq 0.$$

Finally given an initial REE $\{r_t, Y_t\}$ and a new interest rate path $\{\hat{r}_t\}$, the level-k equilibria $\{\hat{r}_t, \hat{Y}^k_t\}$ satisfy the following recursion over $k \geq 0$:

$$\hat{Y}^k_t = (\rho+\lambda)[\int_0^\infty \delta \hat{Y}^{k-1}_{t+s} e^{-\int_0^s r_{t+u}du} ds + \int_0^\infty (1-\delta) \hat{Y}^{k-1}_{t+s} e^{-\int_0^s (r_{t+u}+\lambda)du} ds] \quad \forall t \geq 0.$$

with the initialization that $\hat{Y}^0_t = Y_t$ for all $t \geq 0$.

We now turn to the computation of the different interest rate elasticities of output around a steady state REE $\{R_t, Y_t\}$ $Y_t = Y > 0$ and $r_t = r$ for all $t \geq 0$ with

$$1 = (1-\delta)\frac{\rho+\lambda}{r+\lambda} + \delta\frac{\rho+\lambda}{r}.$$

Later when we derive comparative statics when we vary λ, we vary ρ to keep the interest rate constant at r.

Monetary policy at different horizons under RE. We start with the RE case, where we use the selection that $\lim_{t\to\infty} Y_t = Y$ as we perform the comparative statics underlying the computation of the interest rate elasticities of output.

Proposition 4 (Perpertual youth model of borrowing constraints, RE). *Consider the perpetual youth model of borrowing constraints with logarithmic utility $\sigma = 1$ and rational expectations. For $t > \tau$ the interest rate elasticities of output are zero $\epsilon_{t,\tau} = 0$. For $t \leq \tau$, they depend only on the horizon $\tau - t$ and are given*

$$\epsilon_{t,\tau} = 1.$$

They can be decomposed as $\epsilon_{t,\tau} = \epsilon^{PE}_{t,\tau} + \epsilon^{GE}_{t,\tau}$ into PE and GE elasticities $\epsilon^{PE}_{t,\tau}$ and $\epsilon^{GE}_{t,\tau}$. For $t > \tau$

these elasticities are zero $\epsilon_{t,\tau}^{PE} = \epsilon_{t,\tau}^{GE} = 0$. *For* $t \leq \tau$, *they are given by*

$$\epsilon_{t,\tau}^{PE} = (1-\delta)\frac{\rho+\lambda}{r+\lambda}e^{-(r+\lambda)(\tau-t)} + \delta\frac{\rho+\lambda}{r}e^{-r(\tau-t)}$$
$$\epsilon_{t,\tau}^{GE} = (1-\delta)\frac{\rho+\lambda}{r+\lambda}[1-e^{-(r+\lambda)(\tau-t)}] + \delta\frac{\rho+\lambda}{r}[1-e^{-r(\tau-t)}].$$

A remarkable result in this proposition is that the interest rate elasticity of output $\epsilon_{t,\tau}$ is completely independent of the frequency λ of binding borrowing constraints

$$\frac{\partial \epsilon_{t,\tau}}{\partial \lambda} = 0,$$

and is therefore exactly identical to its counterpart in the representative agent model as described in Proposition 1 adapted to continuous time in Proposition 3. In other words, the incompleteness of markets introduced in the perpetual youth model of borrowing constraints is irrelevant for the aggregate effects of monetary policy. This is essentially a version of the irrelevance-of-incomplete-markets result in Werning (2015). Although the result also holds for $\delta > 0$, the intuition is conveyed most transparently in the case of no outside liquidity $\delta = 0$ because in this case $\rho = r$ is independent of λ, otherwise we have to vary ρ so as to keep r constant when we vary λ. The PE effect is weaker, the higher is λ, so that

$$\frac{\partial \epsilon_{t,\tau}^{PE}}{\partial \lambda} = -(\tau-t)e^{-(r+\lambda)(\tau-t)} < 0.$$

This is because for a given path of output, a higher frequency λ of borrowing constraints leads to more discounting of future interest rate cuts, and hence to a lower response of partial equilibrium consumption to a future interest rate cut. But the GE effect is stronger, the higher is λ, leading to a complete offset

$$\frac{\partial \epsilon_{t,\tau}^{GE}}{\partial \lambda} = -\frac{\partial \epsilon_{t,\tau}^{PE}}{\partial \lambda} > 0.$$

This is because the aggregate marginal propensity to consume $\rho + \lambda = r + \lambda$ and hence the general equilibrium Keynesian multiplier increase with the frequency λ of borrowing constraints.[8]

Monetary policy at different horizons under level-k. We now turn to the level-k case.

[8]Note that this property holds despite the existence of a countervailing effect that arises because the increase in human wealth associated with the general equilibrium increase in output is lower when λ is higher because human wealth is more discounted.

Proposition 5 (Perpertual youth model of borrowing constraints, level-k). *Consider the representative agent model with logarithmic utility $\sigma = 1$ and level-k thinking. For $t > \tau$ the interest rate elasticities of output are zero $\epsilon^k_{t,\tau} = 0$. For $t \leq \tau$, they depend only on the horizon $\tau - t$ and are given by the recursion*

$$\epsilon^k_{t,\tau} = \frac{\delta \frac{e^{-r(\tau-t)}}{r} + (1-\delta)\frac{e^{-(r+\lambda)(\tau-t)}}{r+\lambda} + \delta \int_0^{\tau-t} \epsilon^{k-1}_{t+s,\tau} e^{-rs} ds + (1-\delta) \int_0^{\tau-t} \epsilon^{k-1}_{t+s,\tau} e^{-(r+\lambda)s} ds}{\delta \frac{1}{r} + (1-\delta)\frac{1}{r+\lambda}}.$$

This simplifies in the extreme cases of no outside liquidity $\delta = 0$ and very abundant outside liquidity $\delta \to 1$:

$$\epsilon^k_{t,\tau} = e^{-(r+\lambda)(\tau-t)} \mathcal{E}^k_{ct}((r+\lambda)(\tau-t)) \quad \textit{when} \quad \delta = 0,$$
$$\epsilon^k_{t,\tau} = e^{-r(\tau-t)} \mathcal{E}^k_{ct}(r(\tau-t)) \quad \textit{when} \quad \delta \to 1.$$

Unlike the rational expectations case, level-k, the interest rate elasticity of output $\epsilon^k_{t,\tau}$ depends of the frequency λ of binding borrowing constraints, breaking the irrelevance-of-incomplete-markets result in Werning (2015). Indeed, there are now similarities but also important differences between Proposition 5 and its counterpart in the representative agent model as described in Proposition 2 adapted to continuous time in Proposition 3.

Like in representative agent's case, level-k thinking features (weak) mitigation $\epsilon^k_{t,\tau} < \epsilon_{t,\tau}$, and monotonic convergence with $\frac{\partial \epsilon^k_{t,\tau}}{\partial k} > 0$ and $\lim_{k\to\infty} \epsilon^k_{t,\tau} = 1$. In addition, level-1 coincides exactly with the PE effect $\epsilon^1_{t,\tau} = \epsilon^{PE}_{t,\tau}$.

But $\epsilon^k_{t,\tau}$ depends on the frequency λ of binding borrowing constraints, and as a result differs from its value in the rational expectations case as long as $\delta < 1$, where we vary ρ to keep the interest rate r constant as we vary λ. For simplicity, we focus on the case with no outside liquidity $\delta = 0$ where $r = \rho$, which leads to very transparent formulas. For any k, $\epsilon^k_{t,\tau}$ decreases with λ so that more frequent borrowing constraints lead to stronger *mitigation* of the effects of monetary policy

$$\frac{\partial \epsilon^k_{t,\tau}}{\partial \lambda} = -e^{-(r+\lambda)(\tau-t)} \frac{(r+\lambda)^{k-1}(\tau-t)^k}{(k-1)!} < 0.$$

Moreover, for any k, $\frac{\partial \epsilon^k_{t,\tau}}{\partial \tau}$ decreases with λ so that more frequent borrowing constraints lead to stronger *horizon* effects of monetary policy for small enough horizons

$$\frac{\partial^2 \epsilon^k_{t,\tau}}{\partial \lambda \partial \tau} = e^{-(\rho+\lambda)(\tau-t)} \frac{(r+\lambda)^{k-1}(\tau-t)^{k-1}}{(k-1)!} [(r+\lambda)(\tau-t) - k] < 0 \quad \text{for} \quad (r+\lambda)(\tau-t) < k.$$

These effect disappears in the rational expectations case which obtains in the limit where k goes to ∞. These effects disappear when outside liquidity is very abundant in the limit where δ goes to 1 since then $\epsilon^{k}_{t,\tau} = e^{-r(\tau-t)}\mathcal{E}^{k}_{ct}(r(\tau-t))$ is independent of λ.

This can be seen most clearly in the case $k = 1$ where when $\delta = 0$, we get

$$\epsilon^{1}_{t,\tau} = e^{-(r+\lambda)t}, \quad \text{and} \quad \frac{\partial \epsilon^{1}_{t,\tau}}{\partial \tau} = -(r+\lambda)\epsilon^{1}_{t,\tau},$$

so that the strength of the mitigation and horizon effects is $r + \lambda$ instead of r in the representative agent's case. As a result, the mitigation and horizon effects are plausibly much stronger than in the representative agent's case, even if the interest rate is very low. If the annual interest rate is $r = 5\%$, then the effects of monetary policy decrease at rate 5% per year with a half life of 14 years if $\lambda = 0$ as in the representative agent's case, but decrease at rate 15% per year with a half life of 5 years if $\lambda = 10\%$; if the annual interest rate is 1% the effects of monetary policy decrease at rate 11% per year with a half life of 69 years if $\lambda = 0$ as in the representative agent's case, but decrease at rate 11% per year with a half life of 6 years if $\lambda = 10\%$. In the limit of very abundant outside liquidity instead, we have $\epsilon^{1}_{t,\tau} = e^{-rt}$ and $\frac{\partial \epsilon^{1}_{t,\tau}}{\partial \tau} = -r\epsilon^{1}_{t,\tau}$ as in the representative agent's case and independently of λ.

The results for a finite k are in striking contrast to the rational expectations benchmark, which obtains in the limit where k goes to ∞. Level-k thinking leads to a *mitigation* of the effects of monetary policy so that interest rate changes have less of an effect on output. Level-k thinking also leads to a *horizon* effect of monetary policy so that interest rate changes have less of an effect on output, the further in the future they take place. The mitigation and horizon effects that arise with level-k thinking are stronger, the more frequent are borrowing constraints, i.e. the higher is λ. This illustrates a profound *interaction* between level-k thinking and incomplete markets. This interaction disappears in the limit where outside liquidity is very abundant when δ goes to 1.

There is a simple intuition for all these results in terms of the decomposition of the effects of monetary policy into PE and GE effects. As already explained in Section 3, the PE effect features mitigation—the effect of interest rate changes is lower than the full effect under rational expectations because the latter is the sum of the GE and the PE effect. It also features horizon—for a fixed path of output, interest rate changes affect partial equilibrium consumption less, the further in the future they are. Under rational expectations, the GE effect eliminates the mitigation feature by adding to the PE effect, and eliminates the horizon effect because the GE effect features an anti-horizon effect. At round $k = 1$, monetary policy coincides with the PE effect and features mitigation and horizon, and in the rational expectations limit when k goes to ∞, the mitigation and horizon effects disappear. Intermediate values of k interpolate smoothly and monotonically between these two extremes.

The effects of the frequency λ of binding borrowing constraints can be understood as follows. The horizon and mitigation effects of the PE effect are stronger, the higher is λ because of higher discounting of non-financial (human) wealth. Under rational expectations, the GE effect offsets this dependence on λ because the aggregate marginal propensity to consume $\rho + \lambda$ and hence the Keynesian multiplier increase with λ. At level-1, monetary policy coincides with the PE effect and the horizon and mitigation features are stronger, the higher is λ. In the rational expectations limit where k goes to ∞, the dependence of the mitigation and horizon features on λ disappear. Intermediate values of k interpolate smoothly and monotonically between these two extremes. This also explains why the interaction disappears in the limit where outside liquidity is very abundant when δ goes to 1, since only non-financial (human) wealth is more discounted when borrowing constraints bind more often, but not the dividends promised by the Lucas trees.

5 The Aiyagari-Bewley Model of Borrowing Constraints and Precautionary Savings

In this section we consider a standard Aiyagari-Bewley model of incomplete markets. This model features not only occasionally binding borrowing constraints as the perpetual youth model of borrowing constraints developed in Section 4 but also precautionary savings. As a result, individual consumption functions are no longer linear and are concave instead.

There is a unit mass of infinitely-lived agents indexed by i distributed uniformly over $[0,1]$. Time is discrete, with a period taken to be a quarter. Agents have logarithmic utility $\sigma = 1$ and discount factor β.

Agents face idiosyncratic non-financial income risk $y_t(1-\delta)Y_t$. There is a unit supply of Lucas trees capitalizing the flow of dividends δY_t. The idiosyncratic income process $\log(y_t^i) = \rho \log(y_{t-1}^i) + \epsilon_t^i$ with ϵ_t^i i.i.d. over time and independent across agents is a truncated normal with variance σ_ϵ^2 and such that $\mathbb{E}[\epsilon_t^i] = 1$ so that $\int \epsilon_t^i di = 1$.

Agents can borrow and lend subject to borrowing constraints. We assume that the borrowing contracts have the same form as the Lucas trees. In other words the agents can have negative positions in Lucas trees but the value of their negative position is bounded by a borrowing constraint $B\frac{R}{R_t}Y_t$ where B is a fraction of the natural borrowing limit when the interest rate is at its steady-state value of R. These choices ensure that under rational expectations, the irrelevance result of Werning (2015) holds, and the interest rate elasticity of output coincides with that of a complete markets model (representative agent) $\epsilon_{t,\tau} = 1$.

Individual problem. We first describe the individual problem. We proceed as in Section 3.1 to formulate the individual problem given the aggregate paths $\{Y_t, Y_t^e, R_t\}$ directly in terms of total financial wealth a_t^i as long as Lucas trees satisfy the no-arbitrage conditions (6).

The problem on an at date t with financial wealth a_t^i is therefore

$$\max_{\{\tilde{c}_{t+s}^i, \tilde{a}_{t+1+s}^i\}} \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \log(\tilde{c}_{t+s}^i) ds,$$

subject to the current actual budget constraint

$$\tilde{c}_t^i = (1-\delta) y_t^i Y_t + a_t^i - \frac{\tilde{a}_{t+1}^i}{R_t},$$

and the future expected budget constraints

$$\tilde{c}_{t+1+s}^i = (1-\delta) y_{t+1+s}^i Y_{t+1+s} + \tilde{a}_{t+1+s}^i - \frac{\tilde{a}_{t+2+s}^i}{R_{t+1+s}} \quad \forall s \geq 0.$$

and the borrowing constraints

$$\tilde{a}_{t+s}^i \geq -BRY_t \quad \forall s \geq 0.$$

The policy function for consumption at date t is the individual consumption function and is given by $c^*(a_t^i; \{R_{t+s}\}, Y_t, \{Y_{t+s}^e\})$. The law of motion for a_t^i is given by the actual (as opposed to expected) budget constraints $a_{t+1}^i = R_t[(1-\delta) y_t^i Y_t + a_t^i - c^*(a_t^i; \{R_{t+s}\}, Y_t, \{Y_{t+s}^e\})]$.

Aggregate state variable. The model also features an aggregate state variable as in Section 2.2: the wealth distribution $\Psi_t = \{a_t^i\}$. which is the aggregate state variable for this model. The law of motion for Ψ_t is entirely determined by the laws of motion for individual financial wealth a_t^i given an initial condition Ψ_0 with $\int a_0 d\Psi(a_0) = V_0$, where V_0 is given by the no-arbitrage condition (6). In contrast to the perpetual youth model of borrowing constraints developed in Section 4, this aggregate state variable is required to characterize the aggregate equilibrium.

Reduced-form aggregate consumption function. The reduced-form aggregate consumption function is obtained by aggregating over i the individual consumption function

$$C(\{R_{t+s}\}, Y_t, \{Y^e_{t+s}\}, \Psi_t) = \int_0^1 c^*(a_t; \{r_{t+s}\}, \{Y^e_{t+s}\}) d\Psi_t(a_t).$$

Temporary equilibria, RE equilibria, and level-k equilibria are then defined exactly as in the general reduced form model described in Section 2.

Monetary policy at different horizons. This model cannot be solved analytically. Instead we rely on simulations. We consider a steady state $\{Y, R, \Psi\}$ of the model with a 2% annual interest rate $R = 1.02$. We take $\rho = 0.966$, and $\sigma^2_\epsilon = 0.0107$ for the idiosyncratic income process. For our baseline economy, we take $\frac{V}{Y} = 1.44$ for the fraction of outside liquidity to output, exactly as in McKay et al. (2015), and $B = 0$.[9] The values of $\beta = 0.988$ and $\delta = 0.035$ are calibrated to deliver these values of R and $\frac{V}{Y}$. The fraction of borrowing-constrained agents in the steady state is then 14.7%.

Figure 1 depicts the proportional output response of the economy to a 1% interest rate cut at different horizons, or in other words, the interest rate elasticity of output $\epsilon_{0,\tau}$ at different horizons τ, for the baseline economy. The top panel illustrates the strong mitigation and horizon effects brought about by the interaction of incomplete markets and bounded rationality by comparing the economy with $k = 1$ and incomplete markets to the economy with $k = 1$ and complete markets (representative agent) and the economy with $k = \infty$ (rational expectations) and complete markets (representative agent). The bottom panel shows how these mitigation and horizon effects dissipate as we increase the level of reasoning k moving towards rational expectations $k = \infty$.

Figure 2 illustrates how these effects change as we move away from the baseline economy by varying the discount factor β. These different calibrations lead to different values for the fraction of borrowing-constrained agents in the steady state and for the aggregate marginal propensity to consume. Once again the figure powerfully illustrates the strong interaction of incomplete markets and bounded rationality: For a given finite value of k, the mitigation and horizon effects are much stronger when markets are more incomplete in the sense that the steady-state fraction of borrowing constrained agents is higher.

Overall, in this calibrated Aiyagari-Bewley economy with occasionally borrowing constraints and precautionary savings, there are powerful interactions between bounded rationality and incomplete markets. This reinforces the analytical results that we obtained

[9]This value for the fraction of outside liquidity to output $\frac{V}{Y} = 1.44$ is meant to capture the value of liquid (as opposed to illiquid) wealth in the data.

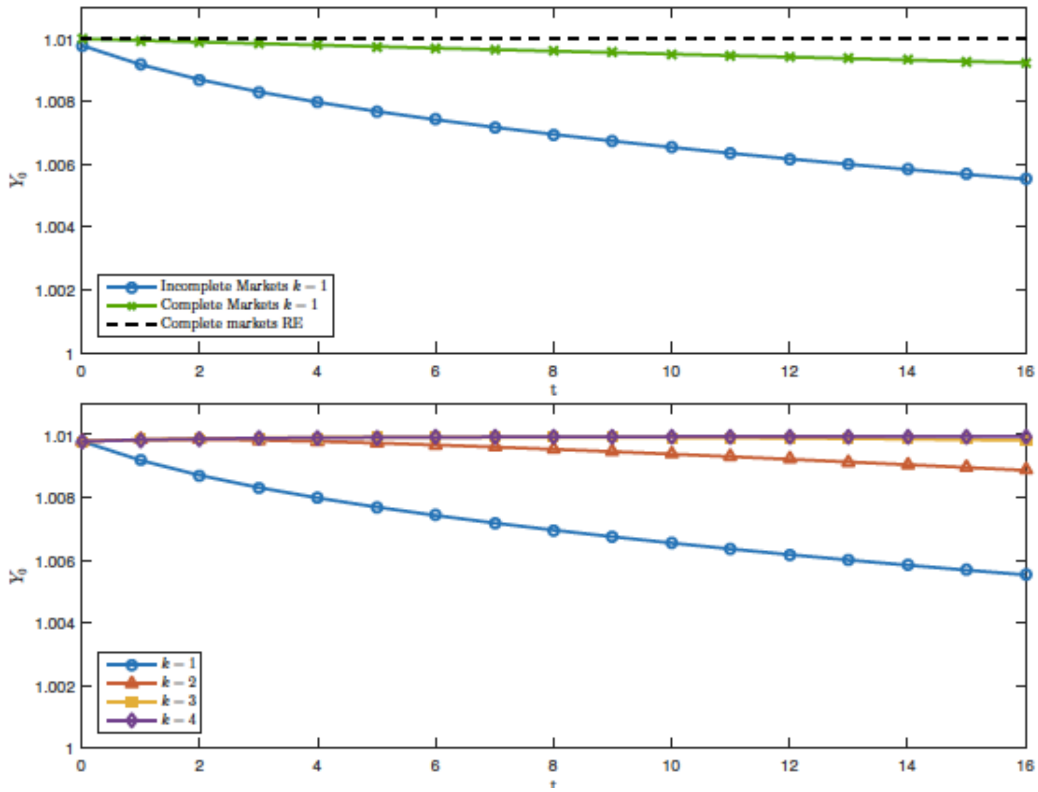

Figure 1: Proportional output response $\epsilon_{0,\tau}$ at date 0 to a 1% interest rate cut at different horizons τ for the baseline economy.

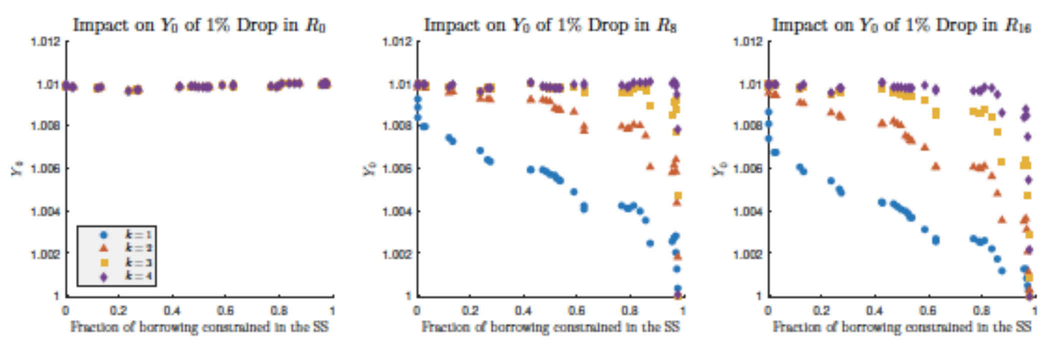

Figure 2: Proportional output response $\epsilon_{0,\tau}$ at date 0 to a 1% interest rate cut at a horizon of $\tau = 0$, $\tau = 8$ quarters, and $\tau = 26$ quarters. Different colors represent equilibrium output under level-k thinking with different values of k. Different dots of the same color correspond to economies with different fractions of borrowing-constrained agents in steady state. This variation is achieved by varying the discount factor β and keeping the steady-state annual interest rate constant at 2%.

in the perpetual youth model of borrowing constraints developed in Section 4 which features borrowing constraints but no precautionary savings.

6 Conclusion

We have demonstrated a strong interaction between two forms of frictions, bounded rationality and incomplete markets. In economies with nominal rigidities, this interaction has important implications for the transmission of monetary policy, by mitigating its effects, the more so, the further in the future that monetary policy change takes place. This offers a possible rationalization of the so-called "forward-guidance puzzle". We conjecture that these conclusions generalize to other shocks and policies, and in economies with less extreme forms of nominal rigidities than the one we have considered here. We leave these issues for future work.

7 Appendix

7.1 Proofs of Propositions 1 and 2

We consider an initial REE $\{R_t, Y_t\}$ which is a steady state with $R_t = R$ and $Y_t = Y$ for all $t \geq 0$. This only requires that $\beta R = 1$. We consider a change $\{\hat{R}_t\}$ in the path for the interest rate ΔR_τ at date τ so that $\hat{R}_\tau = R + \Delta R_\tau$ and $\hat{R}_t = R_t$ for $t \neq \tau$.

We start by computing the new new REE $\{\hat{R}_t, \hat{Y}_t\}$. Because the aggregate model is purely forward looking, we can immediately conclude that for $t > \tau$, $\hat{Y}_t = Y$ and so $\Delta\hat{Y}_t = 0$. And we guess and verify that for $t \leq \tau$, $\hat{Y}_t = Y(1+\frac{\Delta R}{R})^{-\sigma}$ and so

$$\Delta\hat{Y}_t = Y[(1+\frac{\Delta R}{R})^{-\sigma} - 1].$$

This immediately implies that for $t > \tau$, we have $\epsilon_{t,\tau} = 0$ and for $t \leq \tau$, we have

$$\epsilon_{t,\tau} = \sigma.$$

We can perform the decomposition into a partial equilibrium effect and a general equilibrium effect. For $t > \tau$, we have $\Delta\hat{Y}_t^{PE} = \Delta\hat{Y}_t^{GE} = 0$, and for $t \leq \tau$, we have

$$\Delta\hat{Y}_t^{PE} = Y\frac{\frac{(1+\frac{\Delta R}{R})^{-1} - (1+\frac{\Delta R}{R})^{\sigma-1}}{R^{\tau-t+1}}}{1+\frac{(1+\frac{\Delta R}{R})^{\sigma-1}-1}{R^{\tau-t+1}}},$$

$$\Delta\hat{Y}_t^{GE} = Y[(1+\frac{\Delta R}{R})^{-\sigma} - 1] - Y\frac{\frac{(1+\frac{\Delta R}{R})^{-1} - (1+\frac{\Delta R}{R})^{\sigma-1}}{R^{\tau-t+1}}}{1+\frac{(1+\frac{\Delta R}{R})^{\sigma-1}-1}{R^{\tau-t+1}}}.$$

This immediately implies that for $t > \tau$, we have $\epsilon_{t,\tau}^{PE} = \epsilon_{t,\tau}^{GE} = 0$, and for $t \leq \tau$, we have

$$\epsilon_{t,\tau}^{PE} = \sigma\frac{1}{R^{\tau-t+1}},$$

$$\epsilon_{t,\tau}^{GE} = \sigma(1-\frac{1}{R^{\tau-t+1}}).$$

Next we compute the level-k equilibria $\{\hat{R}_t, \hat{Y}_t^k\}$. We have

$$\hat{Y}_t^k = \frac{\sum_{s=0}^{\tau-t-1}\frac{\hat{Y}_{t+1+s}^{k-1}}{R^{1+s}} + (1+\frac{\Delta R}{R})^{-1}\sum_{s=\tau-t}^{\infty}\frac{\hat{Y}_{t+1+s}^{k-1}}{R^{1+s}}}{\frac{1}{R}\frac{1-\frac{1}{R^{\tau-t}}}{1-\frac{1}{R}} + (1+\frac{\Delta R}{R})^{\sigma-1}\frac{\frac{1}{R^{\tau-t+1}}}{1-\frac{1}{R}}}.$$

This implies that

$$\Delta\hat{Y}_t^k = \frac{\sum_{s=0}^{\tau-t-1}\frac{\Delta\hat{Y}_{t+1+s}^{k-1}}{R^{1+s}} + (1+\frac{\Delta R}{R})^{-1}\sum_{s=\tau-t}^{\infty}\frac{\Delta\hat{Y}_{t+1+s}^{k-1}}{R^{1+s}} + Y\frac{(1+\frac{\Delta R}{R})^{-1}-(1+\frac{\Delta R}{R})^{\sigma-1}}{1-\frac{1}{R}}\frac{1}{R^{\tau-t+1}}}{\frac{1}{R}\frac{1-\frac{1}{R^{\tau-t}}}{1-\frac{1}{R}} + \frac{(1+\frac{\Delta R}{R})^{\sigma-1}}{1-\frac{1}{R}}\frac{1}{R^{\tau-t+1}}},$$

from which we get that $\epsilon_{t,\tau}^k$ solves the following recursion over $k \geq 0$:

$$\epsilon_{t,\tau}^k = R(1-\frac{1}{R})\sum_{s=0}^{\infty}\frac{\epsilon_{t+1+s,\tau}^{k-1}}{R^{1+s}} + \sigma\frac{1}{R^{\tau-t}},$$

with the initialization $\epsilon_{t,\tau}^0 = 0$. For $t > \tau$, we have $\epsilon_{t,\tau}^k = 0$. For $t \leq \tau$ we get

$$\epsilon_{t,\tau}^1 = \sigma\frac{1}{R^{\tau-t}},$$

$$\epsilon_{t,\tau}^2 = \sigma\frac{1}{R^{\tau-t}}\left[1+(R-1)(\tau-t)\right],$$

$$\epsilon_{t,\tau}^3 = \sigma\frac{1}{R^{\tau-t}}\left[1+(R-1)(\tau-t)+(R-1)^2\frac{(\tau-t-1)(\tau-t)}{2}\right],$$

and more generally

$$\epsilon_{t,\tau}^k = \sigma\frac{1}{R^{\tau-t}}\left[\sum_{n=0}^{k}(R-1)^n\sum_{s_0=0}^{\tau-t-1}\sum_{s_1=0}^{\tau-t-1-s_0}\cdots\sum_{s_{n-2}=0}^{\tau-t-1-s_{n-3}}1\right].$$

7.2 The Perpetual Youth Model of Borrowing Constraints with $\sigma \neq 1$

Individual consumption function. When $\sigma \neq 1$, the individual consumption function is given by

$$c^*(a_t^i;\{r_{t+s}\},\{Y_{t+s}^e\}) = \frac{a_t^i + \int_0^\infty (1-\delta)Y_{t+s}^e e^{-\int_0^s(r_{t+u}+\lambda)du}ds}{\int_0^\infty e^{-\int_0^s[(1-\sigma)(r_{t+u}+\lambda)+\sigma(\rho+\lambda)]du}ds}.$$

Aggregate state variable. Exactly as in the case $\sigma = 1$ treated in Section 4, the aggregate state variable Ψ_t (the wealth distribution) is not required to characterize the aggregate equilibrium since the reduced-form aggregate consumption function is independent of Ψ_t.

Reduced-form aggregate consumption function. The reduced-form aggregate consumption function is given by

$$C(\{r_{t+s}\},\{Y^e_{t+s}\}) = \frac{\int_0^\infty \delta Y^e_{t+s} e^{-\int_0^s r_{t+u}du} ds + \int_0^\infty (1-\delta) Y^e_{t+s} e^{-\int_0^s (r_{t+u}+\lambda)du} ds}{\int_0^\infty e^{-\int_0^s [(1-\sigma)(r_{t+u}+\lambda)+\sigma(\rho+\lambda)]du} ds}.$$

Equilibrium characterization. For concreteness, we briefly characterize the various equilibria in the context of this particular model. Given beliefs $\{Y^e_t\}$, and given the path for interest rates $\{r_t\}$, $\{r_t, Y_t\}$ is a temporary equilibrium if and only if the path for aggregate income $\{Y_t\}$ is given by

$$Y_t = \frac{\int_0^\infty \delta Y^e_{t+s} e^{-\int_0^s r_{t+u}du} ds + \int_0^\infty (1-\delta) Y^e_{t+s} e^{-\int_0^s (r_{t+u}+\lambda)du} ds}{\int_0^\infty e^{-\int_0^s [(1-\sigma)(r_{t+u}+\lambda)+\sigma(\rho+\lambda)]du} ds} \quad \forall t \geq 0.$$

Similarly, given the path for interest rates $\{r_t\}$, $\{r_t, Y_t\}$ is an REE if and only if the path for aggregate income $\{Y_t\}$ satisfies the fixed point

$$Y_t = \frac{\int_0^\infty \delta Y_{t+s} e^{-\int_0^s r_{t+u}du} ds + \int_0^\infty (1-\delta) Y_{t+s} e^{-\int_0^s (r_{t+u}+\lambda)du} ds}{\int_0^\infty e^{-\int_0^s [(1-\sigma)(r_{t+u}+\lambda)+\sigma(\rho+\lambda)]du} ds} \quad \forall t \geq 0.$$

Finally given an initial REE $\{r_t, Y_t\}$ and a new interest rate path $\{\hat{r}_t\}$, the level-k equilibria $\{\hat{r}_t, \hat{Y}^k_t\}$ satisfy the following recursion over $k \geq 0$:

$$\hat{Y}^k_t = \frac{\int_0^\infty \delta \hat{Y}^{k-1}_{t+s} e^{-\int_0^s r_{t+u}du} ds + \int_0^\infty (1-\delta) \hat{Y}^{k-1}_{t+s} e^{-\int_0^s (r_{t+u}+\lambda)du} ds}{\int_0^\infty e^{-\int_0^s [(1-\sigma)(r_{t+u}+\lambda)+\sigma(\rho+\lambda)]du} ds} \quad \forall t \geq 0.$$

with the initialization that $\hat{Y}^0_t = Y_t$ for all $t \geq 0$.

We now turn to the computation of the different interest rate elasticities of output around a steady state REE $\{R_t, Y_t\}$ $Y_t = Y > 0$ and $r_t = r$ for all $t \geq 0$, where the steady-state interest rate r is given by

$$1 = [(1-\sigma)(r+\lambda) + \sigma(\rho+\lambda)][\frac{\delta}{r} + \frac{1-\delta}{r+\lambda}],$$

so that $r = \rho$ in the limit where the frequency of binding borrowing constraints λ goes to zero.

Monetary policy at different horizons under RE. The interest rate elasticities of output $\epsilon_{t,\tau}$ are zero for $t > \tau$ and otherwise depend only on the horizon $\tau - t$. For $t \leq \tau$, they are the solution of the following integral equation

$$\epsilon_{t,\tau} = [(1-\sigma)(r+\lambda)+\sigma(\rho+\lambda)][\delta \int_0^{\tau-t} \epsilon_{t+s,\tau} e^{-rs} ds + (1-\delta) \int_0^{\tau-t} \epsilon_{t+s,\tau} e^{-(r+\lambda)s} ds]$$
$$+ [(1-\sigma)(r+\lambda)+\sigma(\rho+\lambda)][\delta \frac{e^{-r(\tau-t)}}{r} + (1-\delta)\frac{e^{-(r+\lambda)(\tau-t)}}{r+\lambda}]$$
$$+ (\sigma-1)e^{-[(1-\sigma)(r+\lambda)+\sigma(\rho+\lambda)](\tau-t)}.$$

Define

$$A_s = [(1-\sigma)(r+\lambda)+\sigma(\rho+\lambda)][\delta e^{-rs} + (1-\delta)e^{-(r+\lambda)s}]$$

and

$$B_\tau = [(1-\sigma)(r+\lambda)+\sigma(\rho+\lambda)][\delta \frac{e^{-r\tau}}{r} + (1-\delta)\frac{e^{-(r+\lambda)\tau}}{r+\lambda}] + (\sigma-1)e^{-[(1-\sigma)(r+\lambda)+\sigma(\rho+\lambda)]\tau}.$$

Then the solution is

$$\epsilon_{t,\tau} = \sum_{n=1}^{\infty} \int_0^{\tau-t} A_{\tau-t-s_1} \int_0^{s_1} A_{s_1-s_2} \cdots \int_0^{s_{n-1}} A_{s_{n-1}-s_n} B_{s_n} ds_1 ds_2 \dots ds_n,$$

with the convention that $s_0 = \tau - t$. The PE and GE effects zero for $t > \tau$ and otherwise only depend on the horizon $\tau - t$ and are given by

$$\epsilon^{PE}_{t,\tau} = B_{\tau-t},$$

$$\epsilon^{GE}_{t,\tau} = [(1-\sigma)(r+\lambda)+\sigma(\rho+\lambda)][\delta \int_0^{\tau-t} \epsilon_{t+s,\tau} e^{-rs} ds + (1-\delta) \int_0^{\tau-t} \epsilon_{t+s,\tau} e^{-(r+\lambda)s} ds]$$

with $\epsilon_{t,\tau} = \epsilon^{PE}_{t,\tau} + \epsilon^{GE}_{t,\tau}$.

These expressions can be simplified in three special cases. The first case is when $\sigma = 1$ and is treated in the main body of the paper.

The second case is when the frequency of binding borrowing constraints λ goes to zero, where we get

$$r = \rho,$$

and for for $t \geq \tau$,

$$\epsilon_{t,\tau} = \sigma, \quad \epsilon^{PE}_{t,\tau} = \sigma e^{-r(\tau-t)}, \quad \epsilon^{GE}_{t,\tau} = \sigma[1 - e^{-r(\tau-t)}].$$

The third case is when there is no outside liquidity $\delta = 0$, where we get

$$r = \rho,$$

$$A_s = (r + \lambda)e^{-(r+\lambda)s},$$

$$B_\tau = \sigma e^{-(r+\lambda)\tau},$$

and for $t \geq \tau$,

$$\epsilon_{t,\tau} = \sigma, \quad \epsilon_{t,\tau}^{PE} = \sigma e^{-(r+\lambda)(\tau - t)}, \quad \epsilon_{t,\tau}^{GE} = \sigma[1 - e^{-(r+\lambda)(\tau-t)}].$$

References

García-Schmidt, Mariana and Michael Woodford, "Are Low Interest Rates Deflationary? A Paradox of Perfect-Foresight Analysis," Working Paper 21614, National Bureau of Economic Research October 2015.

Kocherlakota, Narayana R., "Bubbles and constraints on debt accumulation," *Journal of Economic Theory*, 1992, *57* (1), 245–256.

McKay, Alisdair, Emi Nakamura, and Jón Steinsson, "The Power of Forward Guidance Revisited," Working Paper 20882, National Bureau of Economic Research January (revised July 15) 2015.

Werning, Iván, "Incomplete Markets and Aggregate Demand," Working Paper 21448, National Bureau of Economic Research August 2015.

Woodford, Michael, "Public Debt as Private Liquidity," *American Economic Review*, May 1990, *80* (2), 382–88.

_ , "Macroeconomic Analysis Without the Rational Expectations Hypothesis," *Annual Review of Economics*, 2013, *5* (1), 303–346.